

Multiplier Sequences and Real-rootedness of Local h -polynomials of Cluster Subdivisions

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Abstract. In this paper, we prove a conjecture of Athanasiadis that the local h -polynomials of type A cluster subdivisions have only real zeros. We also show the real-rootedness of local h -polynomials of all the other types. Our proof mainly involves multiplier sequences and Chebyshev polynomials of the second kind.

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1 Introduction

The notion of local h -polynomials was introduced by Stanley [13] during his breathtaking study on the face enumeration of subdivisions of simplicial complexes. Let V be an n -element vertex set. Given a subdivision Γ of the abstract simplex 2^V , the local h -polynomial $\ell(\Gamma, x)$ is defined as an alternating sum of the h -polynomials of the restrictions of Γ to the faces of 2^V , namely,

$$\ell_V(\Gamma, x) = \sum_{F \subset V} (-1)^{n-|F|} h(\Gamma_F, x),$$

where $h(\Gamma_F, x)$ is the h -polynomial of Γ_F . Stanley [13] showed that $\ell_V(\Gamma, x)$ has nonnegative and symmetric coefficients, hence the local h -polynomial can be expressed as

$$\ell_V(\Gamma, x) = \sum_{i=0}^{\lfloor n/2 \rfloor} \xi_i x^i (1+x)^{n-2i}. \quad (1)$$

Athanasiadis [1] made an excellent survey on this topic. He [1, Section 4] also gave interesting examples of local h -polynomials for several families of subdivisions and proposed an open question whether these polynomials have only real zeros. One example is about the barycentric subdivision of the simplex 2^V , denoted $\text{sd}(2^V)$. Stanley [13, Proposition 2.4] first showed that the local h -polynomial of $\text{sd}(2^V)$ is the derangement polynomial of order n , first studied by Brenti [4]. The unimodality of this class of polynomials can be

proved in various ways (see [4], [14], [2, Section 4] and [12]), while their real-rootedness was first established by Zhang [15].

This paper is concerning cluster subdivisions. Let I be an n -element set and $\Phi = \{a_i : i \in I\}$ be a root system. The cluster complex $\Delta(\Phi)$, studied by Fomin and Zelevinsky [7, 8], is a simplicial complex on the vertex set of positive roots and negative simple roots. The positive cluster complex $\Delta^+(\Phi)$ is the restriction of $\Delta(\Phi)$ on the positive roots. It naturally defines a geometric subdivision of the simplex on the vertex set of simple roots of Φ , the so-called *cluster subdivision* $\Gamma(\Phi)$. The local h -polynomial $\ell_I(\Gamma(\Phi), x)$ is given by

$$\ell_I(\Gamma(\Phi), x) = \sum_{J \subset I} (-1)^{|I \setminus J|} h(\Delta_+(\Phi_J), x),$$

where Φ_J is the parabolic root subsystem of Φ with respect to J . Although simple closed form expressions for the local h -polynomials of type A and type B were not found until now, the following result of Athanasiadis and Savvidou [2] gave an explicit expression of the numbers $\xi_i(\Phi)$.

Lemma 1.1 ([2]). *Let Φ be an irreducible root system of rank n and Cartan-Killing type \mathcal{X} and let $\xi_i(\Phi)$ be the integers uniquely defined by (1). Then $\xi_0(\Phi) = 0$ and*

$$\xi_i(\Phi) = \begin{cases} \frac{1}{n-i+1} \binom{n}{i} \binom{n-i-1}{i-1}, & \text{if } \mathcal{X} = A_n \\ \binom{n}{i} \binom{n-i-1}{i-1}, & \text{if } \mathcal{X} = B_n \\ \frac{n-2}{i} \binom{2i-2}{i-1} \binom{n-2}{2i-2}, & \text{if } \mathcal{X} = D_n \end{cases}$$

for $1 \leq i \leq \lfloor n/2 \rfloor$. Moreover,

$$\sum_{i=0}^{\lfloor n/2 \rfloor} \xi_i(\Phi) x^i = \begin{cases} (m-2)x, & \text{if } \mathcal{X} = I_2(m) \\ 8x, & \text{if } \mathcal{X} = H_3 \\ 42x + 40x^2, & \text{if } \mathcal{X} = H_4 \\ 10x + 9x^2, & \text{if } \mathcal{X} = F_4 \\ 7x + 35x^2 + 13x^3, & \text{if } \mathcal{X} = E_6 \\ 16x + 124x^2 + 112x^3, & \text{if } \mathcal{X} = E_7 \\ 44x + 484x^2 + 784x^3 + 120x^4, & \text{if } \mathcal{X} = E_8. \end{cases}$$

Athanasiadis [1] conjectured that the local h -polynomials of type A cluster subdivision have only real zeros. In this paper, we confirm his conjecture by multiplier sequences. We also show that the real-rootedness is true for all the other types.

Theorem 1.2. *For any irreducible root system, the local h -polynomial of the cluster subdivision of the simplex has only real zeros.*

The remainder of this paper is organized as follows. In Section 2, we gave an overview of the theory of multiplier sequences. In Section 3, we present our proof of Theorem 1.2.

2 Preliminaries

In this section, we make some necessary preliminaries on real-rooted polynomials. Especially, some facts about multiplier sequences are given.

A sequence of real numbers $\{\lambda_k\}_{k=0}^{\infty}$ is a multiplier sequence, if for every polynomial $\sum_{k=0}^n a_k z^k$ with all zeros real, the polynomial $\sum_{k=0}^n \lambda_k a_k z^k$ is either identically zero or has only real zeros. In the following, we shall list some related facts which will be used in the paper. For a complete introduction of multiplier sequences, we refer the reader to [5, 6, 11].

A fundamental theorem about multiplier sequences is due to Pólya and Schur [10]. Before introducing their result, we recall the notion of Laguerre–Pólya class. An entire function $\phi(x) = \sum_{k=0}^{\infty} \gamma_k \frac{x^k}{k!}$ is in the *Laguerre–Pólya class*, written $\phi \in \mathcal{L}\text{-}\mathcal{P}$ if it can be written as

$$\phi(x) = cx^m e^{-ax^2+bx} \prod_{k=1}^{\omega} \left(1 + \frac{x}{x_k}\right) e^{-\frac{x}{x_k}}, \quad (0 \leq \omega \leq \infty),$$

where $b, c, x_k \in \mathbb{R}$, m is a non-negative integer, $a \geq 0$, $x_k \neq 0$ and $\sum_{k=1}^{\omega} \frac{1}{x_k^2} < \infty$. Let $\mathcal{L}\text{-}\mathcal{P}^+$ denote the set of functions in the Laguerre–Pólya class with nonnegative coefficients, and $\mathcal{L}\text{-}\mathcal{P}(-\infty, 0]$ denote the set of functions in the Laguerre–Pólya class that have only non-positive zeros. A remarkable property is that an entire function is in the Laguerre–Pólya class if and only if it is a locally uniform limit of real polynomials which have only real zeros.

A complete characterization of multiplier sequences was given by Pólya and Schur [10].

Theorem 2.1 (Pólya–Schur). *Let $\{\gamma_k\}_{k=0}^{\infty}$ be a sequence of real numbers. The following statements are equivalent:*

- (i) $\{\gamma_k\}_{k=0}^{\infty}$ is a multiplier sequence;
- (ii) For any non-negative integer n , the polynomial $\sum_{k=0}^n \binom{n}{k} \gamma_k x^k$ has only real zeros of the same sign or are identically zero.
- (iii) Either $\sum_{k=0}^{\infty} \gamma_k \frac{x^k}{k!}$ or $\sum_{k=0}^{\infty} (-1)^k \gamma_k \frac{x^k}{k!}$ belongs to $\mathcal{L}\text{-}\mathcal{P}^+$.

By Theorem 2.1, we obtain the following result.

Lemma 2.2. *For any positive integer n , the sequence $\{\frac{1}{(n-k)!}\}_{k=0}^{\infty}$ is a multiplier sequence.*

Proof. Clearly, the function

$$\sum_{k=0}^{\infty} \frac{1}{(n-k)!} \frac{x^k}{k!} = \frac{1}{n!} (1+x)^n$$

has only real zeros. This completes the proof by Theorem 2.1. \square

The following result of Laguerre can produce several multiplier sequences.

Theorem 2.3 ([6]). *If $\phi(x) \in \mathcal{L}\text{-}\mathcal{P}(-\infty, 0]$, then $\{\phi(k)\}_{k=0}^{\infty}$ is a multiplier sequence.*

Since

$$\frac{1}{\Gamma(x)} = x \exp(\gamma x) \prod_{n=1}^{\infty} \left(1 + \frac{x}{n}\right) \exp\left(-\frac{x}{n}\right)$$

belongs to $\mathcal{L}\text{-}\mathcal{P}(-\infty, 0]$, it follows from Theorem 2.3 that

Lemma 2.4. *The sequence $\{\frac{1}{k!}\}_{k=0}^{\infty} = \{\frac{1}{\Gamma(k+1)}\}_{k=0}^{\infty}$ is a multiplier sequence.*

We now give two multiplier sequences, which will be used in the next section.

Lemma 2.5. *For any positive integer n , the sequence $\{\frac{1}{i!(n-i)!}\}_{i=0}^{\infty}$ is a multiplier sequence.*

Proof. By the definition, the Hadamard product (termwise product) of two multiplier sequences is also a multiplier sequence. Hence, by Lemma 2.2 and Lemma 2.4, it follows that $\{\frac{1}{i!(n-i)!}\}_{i=0}^{\infty}$ is a multiplier sequence. This completes the proof. \square

Before ending this section, we address the following elementary but useful fact.

Lemma 2.6. *The polynomial*

$$\ell(x) = \sum_{i=1}^{n-1} \ell_i x^i = \sum_{i=1}^{\lfloor n/2 \rfloor} \xi_i x^i (1+x)^{n-2i}$$

has only real zeros if and only if the polynomial

$$\xi(x) = \sum_{i=1}^{\lfloor n/2 \rfloor} \xi_i x^i$$

does.

Proof. We first show the necessity. Suppose that the polynomial $\xi(x)$ has only real zeros. Clearly,

$$\ell_V(x) = (1+x)^n \xi\left(\frac{x}{(1+x)^2}\right).$$

Resolving the equation

$$y = \frac{x}{(1+x)^2},$$

we obtain that

$$x = \frac{1 - 2y \pm \sqrt{1 - 4y}}{2y} = \frac{1 \pm \sqrt{1 - 4y}}{2y} - 1.$$

Based on this fact, a routine computation shows that every negative zero of $\xi(x)$ corresponds to a pair of negative zeros of $\ell(x)$, separated by -1 . Since $\xi(x)$ has $\lfloor n/2 \rfloor - 1$ negative zeros in total, it follows that the intervals $(-\infty, -1)$ and $(-1, 0)$ contain respectively $\lfloor n/2 \rfloor - 1$ zeros of $\ell(x)$. Besides, 0 is a zero of $\ell(x)$ and -1 is a zero of the same polynomial if n is odd. Counting all the zeros discussed above, we obtain that $\ell(x)$ has $n - 1$ real zeros. This completes the proof of the necessity.

Conversely, the proof of the sufficiency is similar. This completes the proof. \square

3 Real-rootedness of local h -polynomials of cluster subdivisions

In this section, we shall give our proof of Theorem 1.2 case by case. By Lemma 1.1, one can easily check the local h -polynomials for the exceptional groups have only real zeros. In the following, we shall discuss the case of type A , type B and type D , respectively.

3.1 Type A

In this subsection we deal with the real-rootedness of

$$\ell(\Gamma(A_n), x) = \sum_{i=1}^{\lfloor n/2 \rfloor} \frac{1}{n-i+1} \binom{n}{i} \binom{n-i-1}{i-1} x^i (1+x)^{n-2i}.$$

The main result of this subsection is as follows.

Theorem 3.1. *For any positive integer n , the polynomial $\ell(\Gamma(A_n), x)$ has only real zeros.*

With the aid of Lemma 2.6, we turn our attention to the following polynomial

$$P_n(x) = \sum_{i=1}^{\lfloor n/2 \rfloor} \frac{1}{n-i+1} \binom{n}{i} \binom{n-i-1}{i-1} x^i. \quad (2)$$

We first consider the real-rootedness of the following polynomial

$$\sum_{i=1}^{\lfloor n/2 \rfloor} \binom{n-i-1}{i-1} x^i.$$

Since

$$\sum_{i=1}^{\lfloor n/2 \rfloor} \binom{n-i-1}{i-1} x^i = \sum_{i=1}^{\lfloor n/2 \rfloor} \binom{n-2-(i-1)}{i-1} x^i = x \sum_{j=0}^{\lfloor n/2 \rfloor - 1} \binom{n-2-j}{j} x^j,$$

we focus on the following polynomial

$$H_n(x) = \sum_{j=0}^{\lfloor n/2 \rfloor} \binom{n-j}{j} x^j.$$

Lemma 3.2. *For any positive integer n , the polynomial $H_n(x)$ has only negative and simple zeros.*

Proof. This polynomial is closed related to the Chebyshev polynomial of the second kind

$$U_n(y) = \sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^k \binom{n-k}{k} (2y)^{n-2k}.$$

By replacing y by $1/2y$, it follows that

$$y^n U_n\left(\frac{1}{2y}\right) = \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n-k}{k} (-y^2)^k. \quad (3)$$

From their trigonometric definition, the Chebyshev polynomials of the second kind satisfy

$$U_n(\cos \theta) = \frac{\sin(n+1)\theta}{\sin \theta}.$$

It is known that $\cos(\frac{k}{n+1}\pi)$, where $k = 1, 2, \dots, n$, are the zeros of $U_n(y)$. Since they form distinct pairs of opposite numbers. It follows from (3) that $-\frac{1}{4} \sec^2(\frac{k}{n+1}\pi)$, where $k = 1, 2, \dots, \lfloor n/2 \rfloor$, are the zeros of $H_n(x)$. Therefore, the zeros of $H_n(x)$ are real and simple. This completes the proof. \square

Now we are able to prove Theorem 3.1.

Proof of Theorem 3.1. By Lemma 3.2, the polynomial

$$\sum_{i=1}^{\lfloor n/2 \rfloor} \binom{n-i-1}{i-1} x^i = x H_{n-2}(x)$$

has only real zeros. By Lemma 2.5, the sequence $\{\frac{1}{i!(n-i+1)!}\}_{i \geq 0}$ is a multiplier sequence. Hence,

$$P_n(x) = \sum_{i=1}^{\lfloor n/2 \rfloor} \frac{n!}{i!(n-i+1)!} \binom{n-i-1}{i-1} x^i.$$

has only real zeros. By Lemma 2.6 we obtain that $\ell_I(\Gamma(A_n), x)$ has only real zeros. This completes the proof. \square

3.2 Type B

We now consider the real-rootedness of the following polynomials

$$\ell_I(\Gamma(B_n), x) = \sum_{i=1}^{\lfloor n/2 \rfloor} \binom{n}{i} \binom{n-i-1}{i-1} x^i (1+x)^{n-2i}$$

for any positive integer n .

Along similar lines of the proof of Theorem 3.1, one can show that

Theorem 3.3. *For any positive integer n , the polynomial $\ell_I(\Gamma(B_n), x)$ has only real zeros.*

Proof. By Lemma 3.2, the polynomial

$$\sum_{i=1}^{\lfloor n/2 \rfloor} \binom{n-i-1}{i-1} x^i = x H_{n-2}(x)$$

has only real zeros. By Lemma 2.5, the sequence $\{\frac{1}{i!(n-i)!}\}_{i \geq 0}$ is a multiplier sequence. For any positive integer n , the polynomial

$$\sum_{i=1}^{\lfloor n/2 \rfloor} \binom{n}{i} \binom{n-i-1}{i-1} x^i = n! \sum_{i=1}^{\lfloor n/2 \rfloor} \frac{1}{i!(n-i)!} \binom{n-i-1}{i-1} x^i$$

has only real zeros. By Lemma 2.6 we obtain that $\ell_I(\Gamma(B_n), x)$ has only real zeros. \square

3.3 Type D

We now consider the real-rootedness of the following polynomial

$$\ell_I(\Gamma(D_n), x) = \sum_{i=1}^{\lfloor n/2 \rfloor} \frac{n-2}{i} \binom{2i-2}{i-1} \binom{n-2}{2i-2} x^i (1+x)^{n-2i}$$

for any positive integer n .

This polynomial is a multiple scalars of the *Narayana polynomial*, which has been known to be real-rooted, see [3, 9]. In fact, from the following identity [3]

$$\sum_{i=0}^{\lfloor n/2 \rfloor} \frac{1}{i+1} \binom{2i}{i} \binom{n}{2i} x^i (1+x)^{n-2i} = \sum_{i=0}^n \frac{1}{n+1} \binom{n+1}{i} \binom{n+1}{i+1} x^i,$$

we obtain that

$$\begin{aligned} \sum_{i=1}^{\lfloor n/2 \rfloor} \frac{n-2}{i} \binom{2i-2}{i-1} \binom{n-2}{2i-2} x^i (1+x)^{n-2i} &= (n-2)x \sum_{i=0}^{\lfloor (n-2)/2 \rfloor} \frac{1}{i+1} \binom{2i}{i} \binom{n-2}{2i} x^i (1+x)^{n-2-2i} \\ &= (n-2)x \sum_{i=0}^{n-2} \frac{1}{n-1} \binom{n-1}{i} \binom{n-1}{i+1} x^i. \end{aligned}$$

Hence, we are able to derive the following result.

Theorem 3.4. *For any positive integer $n \geq 2$, the polynomial $\ell_I(\Gamma(D_n), x)$ has only real zeros.*

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